A computation using Stein's Method & Exchangeable Pairs.

Let $W = \sum_{i=1}^n \xi_i$ $\quad$ ξ_i indep. $\quad \mathbb{E}\,\xi_i = 0$

$$\mathbb{E}\sum_{i=1}^n \xi_i^2 = 1$$

Pick I uniformly in $\{1, \dots, n\}$

Let $\{\xi_i^*,\ 1 \le i \le n\}$ be an independent copy of $\{\xi_i,\ 1 \le i \le n\}$.

Let $W' = W - \xi_I + \xi_I^*$

(W', W) is an exchangeable pair.

(W' is obtained by removing one of the ξ_i and replacing it with an indep. copy)

$$\mathbb{E}\big((W'-W) \mid W\big) = \mathbb{E}\big(\mathbb{E}(W'-W \mid \xi_1, \dots, \xi_n) \mid W\big)$$

$$= \mathbb{E}\Big(\tfrac{1}{n}\sum \mathbb{E}\big((\xi_i^* - \xi_i) \mid \xi_1, \dots, \xi_n\big) \mid W\Big)$$

$$= \mathbb{E}\Big(\tfrac{1}{n}\sum(-\xi_i) \mid W\Big) = -\frac{W}{n}$$

So $\lambda = -\frac{1}{n}$

Also, $\mathbb{E}\big((W'-W)^2 \mid W\big) = \frac{1}{n}\Big(1 + \sum_{i=1}^n \mathbb{E}(\xi_i^2 \mid W)\Big)$

(recall $\mathbb{E}\sum(\xi_i^*)^2 = 1$)

Therefore,

$$\mathbb{E}\left|1 - \frac{1}{2\lambda}\mathbb{E}\big((W'-W)^2 \mid W\big)\right| = \mathbb{E}\left|1 - \frac{1}{2}\Big(1 + \sum \mathbb{E}(\xi_i^2 \mid W)\Big)\right|$$

$$= \frac{1}{2}\,\mathbb{E}\left|1 - \sum_i \mathbb{E}(\xi_i^2 \mid W)\right|$$

$$\le \frac{1}{2}\,\mathbb{E}\left|\sum(\xi_i^2 - \mathbb{E}\,\xi_i^2)\right| = R_2$$

Also,

$$\mathbb{E}|W-W'|^3 = \frac{1}{n}\sum \mathbb{E}|\xi_i - \xi_i^*|^3 \le \frac{8}{n}\sum_{i=1}^{n} \mathbb{E}|\xi_i|^3 = R_3$$

We want to show $\mathbb{E}\, Tf(W) \approx \mathbb{E}\left[\frac{f(W')-f(W)}{\lambda}\right]$

$$0 = \mathbb{E}\left[\frac{f(W')-f(W)}{\lambda}\right]$$ (by exchangeability of (W',W) and antisymmetry)

$$= \mathbb{E}\ \frac{1}{\lambda} f'(W)(W'-W) + \frac{1}{\lambda}\frac{f''(W)}{2}(W'-W)^2 + \frac{1}{\lambda} R_{em}$$

where $|R_{em}| \le \frac{\|f'''(W)\|}{3!}|W'-W|^3 \le \frac{\|f'''(W)\|}{3!} R_3$

Condition on W:

$$\mathbb{E}\ \frac{1}{\lambda} f'(W)(W'-W) = \mathbb{E}\left(f'(W)\ \mathbb{E}\left(\frac{W'-W}{\lambda}\Big| W\right)\right)$$

$$= -\mathbb{E}(f'(W)\,W)$$ since $\mathbb{E}(W'-W|W) = -\lambda W$, $\lambda = 1/n$

$$\mathbb{E}\ \frac{1}{\lambda}\frac{f''(W)}{2}(W'-W)^2 = \mathbb{E}\left[f''(W) - f''(W)\left(1 - \frac{1}{2\lambda}|W-W'|^2\right)\right]$$

$$= \mathbb{E} f''(W) - R_4$$

where $|R_4| \le \|f''(W)\|\ \mathbb{E}\left|1 - \frac{1}{2\lambda}\mathbb{E}((W-W')^2|W)\right|$

$\le \|f''(W)\|\ R_2$ (see definition of R_2)

$$\Rightarrow \left|\mathbb{E}\left(f''(W) - Wf'(W)\right)\right| \le \|f''\|\,R_2 + \frac{\|f'''\|}{3!}R_3$$

Note: if $\xi_i = n^{-1/2}X_i$, these remainders will be small $O(n^{-1/2})$
with X_i iid